Code No. ______________________________

Mathematics Ph.D. Qualifier Exam

Department of Mechanical Engineering
Michigan State University

August 2019

Directions: **Open Book (one book allowed), open notes (limited to one 3 ring binder of notes in the student's own handwriting – no photocopies), Calculators NOT permitted**.

All problems carry equal weight.

Exam prepared by Profs. S. Baek and H. Modares.

Problem #1. The Gram-Schmidt orthogonalization is the way of forming mutually orthogonal vectors from a series of given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, … , $\mathbf{v}_m$ (see **Appendix A** in the next page for the Gram-Schmidt orthogonalization). Similarly, the Gram-Schmidt orthogonalization can be applied to a space of functions. Using the orthogonalization process, determine orthogonalized polynominals on a closed interval [-1, 1] (i.e., from the given polynomials, 1, x, x^2, x^3, and x^4, determine orthogonal polynomials up to fourth order). Use the inner product defined as

$$< f \cdot g >= \int_{-1}^{1} f(x)g(x)dx.$$

Appendix A

Gram-Schmidt process:

We define the projection operator by

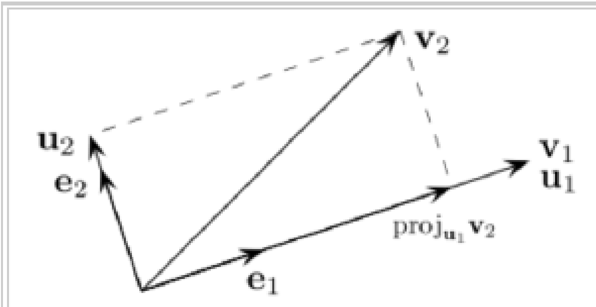
$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes the inner product of the vectors **u** and **v**. This operator projects the vector **v** orthogonally onto the line spanned by vector **u**.

The Gram–Schmidt process then works as follows:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
&\vdots & &\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

The sequence $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is the required system of orthogonal vectors, and the normalized vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k$ form an ortho*normal* set. The calculation of the sequence $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is known as *Gram–Schmidt orthogonalization*, while the calculation of the sequence $\mathbf{e}_1, \ldots, \mathbf{e}_k$ is known as *Gram–Schmidt orthonormalization* as the vectors are normalized.

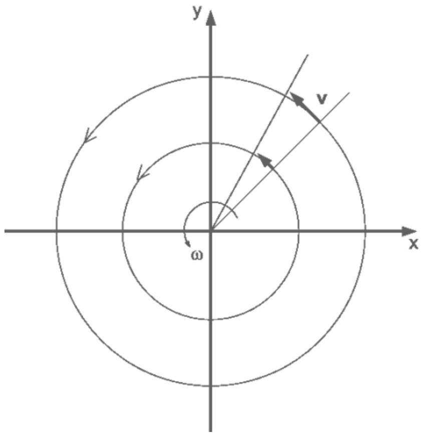


The first two steps of the Gram–Schmidt process

To check that these formulas yield an orthogonal sequence, first compute $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ by substituting the above formula for $\mathbf{u}_2$: we get zero. Then use this to compute $\langle \mathbf{u}_1, \mathbf{u}_3 \rangle$ again by substituting the formula for $\mathbf{u}_3$: we get zero. The general proof proceeds by mathematical induction.

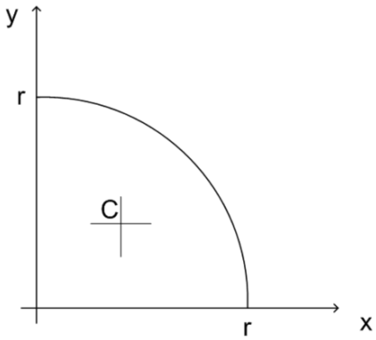
Problem #2. Consider a counterclockwise plane rigid-body rotation about the z axis with an angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$, where ω is constant. Determine the curl of the velocity of a plane rigid rotation (be careful for determining the sign!).

Note: curl $\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}$.



Problem #3. Determine the centroid $C(\bar{x}, \bar{y})$, and the polar moment of inertia (J) with respect to the centroid for a quarter semicircle with the radius r shown in the following figure, where

$$\bar{x} = \frac{1}{A}\int x dA,\ \ \bar{y} = \frac{1}{A}\int y dA,\ \ J = \int (x-\bar{x})^2 + (y-\bar{y})^2\ dA.$$



Problem #4. Find the general solution for the following differential equation

$$\ddot{y} + y = \frac{1}{2}Sec(x)$$

Problem # 5. Consider the following Matrix.

$$\begin{bmatrix} 3 & 6 & -9 \\ 2 & 4 & -5 \\ -2 & -3 & 4 \end{bmatrix}$$

Find the inverse and rank of the matrix.

Problem # 6. Consider the following system of equations:

$$\begin{aligned} 3x_1 + 6x_2 + 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

Use elementary row operations to reduce the augmented matrix corresponding to this system to a Reduced Row Echelon Form (RREF), and then find all its solutions.

Solution: