All work and all steps must be shown, if all work is not provided credit will not be given.

Student Code Number_____

Department of Mechanical Engineering
Michigan State University

Mathematics
Ph.D. Qualifying Exam

August 2018

Closed book, notes and a non-networked calculator
You may use a one page (8.5x11) – **one sided** formula sheet.
All four questions are weighted equally.

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LAPLACE TABLES PROVIDED

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1. Use Laplace Transforms to solve an Initial Value Problem: Solve the following initial value problem using Laplace transforms. Note that $\delta(t - t_o)$ is the Dirac delta function and $u_o(t - t_o)$ is the Heaviside step function.

$$\ddot{y} + y = 3\delta(t - \pi) + 5u_o(t - 3\pi),$$
$$y(0) = \frac{1}{10},\ \dot{y}(0) = \frac{3}{2}.$$

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2. We examine the Laplace and Poisson equations:

   a) Solve the following Laplace equation with associated boundary conditions on the domain $0 < x < 1, 0 < y < 1$.

$$\nabla^2 u = 0,$$

$$\begin{aligned} u(0, y) &= 0, \\ u(1, y) &= 1, \\ u(x, 0) &= 0, \\ u(x, 1) &= 0. \end{aligned}$$

   Find the complete solution including Fourier coefficients, if applicable. Also, discuss any problems or difficulties near the points $(x, y) = (1,0), (1,1)$. Draw the solutions (lines of constant $u(x, y)$)near these points.

   b) Change the *Poisson equation*, $\nabla^2 u = 1$, subject to $u = 0$ on all four sides, into the <u>Laplace equation</u> $\nabla^2 v = 0$ subject to appropriate inhomogeneous boundary conditions on all four sides.

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3. Find all the eigenvalues and an eigenvector for each eigenvalue for the following matrix. Show all steps and annotate each step with words describing what you are doing and why.

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

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4. Solve the system of linear equations for the set of points that satisfies all three equations. The preferred solution approach is to place the equations in augmented matrix form and provide notation on what you are doing for each step.

$$2x_1 + 6x_2 + x_3 = 7$$

$$x_1 + 2x_2 - x_3 = -1$$

$$5x_1 + 7x_2 - 4x_3 = 9$$

Table of Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1. 1	$\frac{1}{s}$
2. t	$\frac{1}{s^2}$
3. t^n	$\frac{n!}{s^{n+1}}$, n positive integer
4. $t^{-1/2}$	$\sqrt{\frac{\pi}{s}}$
5. $t^{1/2}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
6. t^{α}	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$
7. $\sin kt$	$\frac{k}{s^2+k^2}$
8. $\cos kt$	$\frac{s}{s^2+k^2}$
9. $\sin^2 kt$	$\frac{2k^2}{s(s^2+4k^2)}$
10. $\cos^2 kt$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
11. e^{at}	$\frac{1}{s-a}$
12. $\sinh kt$	$\frac{k}{s^2-k^2}$
13. $\cosh kt$	$\frac{s}{s^2-k^2}$
14. $\sinh^2 kt$	$\frac{2k^2}{s(s^2-4k^2)}$
15. $\cosh^2 kt$	$\frac{s^2-2k^2}{s(s^2-4k^2)}$
16. $e^{at}t$	$\frac{1}{(s-a)^2}$
17. $e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$, n a positive integer
18. $e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
19. $e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
20. $e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$
21. $e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$
22. $t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
23. $t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
24. $\sin kt + kt\cos kt$	$\frac{2ks^2}{(s^2+k^2)^2}$
25. $\sin kt - kt\cos kt$	$\frac{2k^3}{(s^2+k^2)^2}$
26. $t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
27. $t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
28. $\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$
29. $\frac{ae^{at}-be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
30. $1-\cos kt$	$\frac{k^2}{s(s^2+k^2)}$
31. $kt-\sin kt$	$\frac{k^3}{s^2(s^2+k^2)}$
32. $\cos at - \cos bt$	$\frac{s(b^2-a^2)}{(s^2+a^2)(s^2+b^2)}$
33. $\sin kt\sinh kt$	$\frac{2k^2s}{s^4+4k^4}$
34. $\sin kt\cosh kt$	$\frac{k(s^2+2k^2)}{s^4+4k^4}$
35. $\cos kt\sinh kt$	$\frac{k(s^2-2k^2)}{s^4+4k^4}$
36. $\cos kt\cosh kt$	$\frac{s^3}{s^4+4k^4}$
37. $\delta(t)$	1

38. $\delta(t-a)$	e^{-as}
39. $\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$
40. $J_0(kt)$	$\dfrac{1}{\sqrt{s^2+k^2}}$
41. $\dfrac{e^{bt}-e^{at}}{t}$	$\ln\dfrac{s-a}{s-b}$
42. $\dfrac{2(1-\cos at)}{t}$	$\ln\dfrac{s^2+a^2}{s^2}$
43. $\dfrac{2(1-\cosh at)}{t}$	$\ln\dfrac{s^2-a^2}{s^2}$
44. $\dfrac{\sin at}{t}$	$\arctan\left(\dfrac{a}{s}\right)$
45. $\dfrac{\sin at\cos bt}{t}$	$\dfrac{1}{2}\arctan\dfrac{a+b}{s}+\dfrac{1}{2}\arctan\dfrac{a-b}{s}$
46. $\dfrac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
47. $\dfrac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
48. $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$
49. $2\sqrt{\dfrac{t}{\pi}}e^{-a^2/4t}-a\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
50. $e^{ab}e^{b^2t}\operatorname{erfc}\left(b\sqrt{t}+\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s}+b)}$
51. $-e^{ab}e^{b^2t}\operatorname{erfc}\left(b\sqrt{t}+\dfrac{a}{2\sqrt{t}}\right)+\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{be^{-\sqrt{s}}}{s(\sqrt{s}+b)}$
52. $e^{at}f(t)$	$F(s-a)$
53. $f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
54. $g(t)\mathcal{U}(t-a)$	$e^{-as}\mathcal{L}\{g(t+a)\}$
55. $f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
56. $t^nf(t)$	$(-1)^n\dfrac{d^n}{ds^n}F(s)$
57. $\displaystyle\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$