Code Number :..............

MATHEMATICS QUALIFYING EXAM

August 2008

OPEN BOOK (only one book allowed)

Answer all questions

All questions have equal weight

TIME: 3.0 hrs

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Problem #1

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 3 & 2i \\ -2i & 0 \end{bmatrix}$$

and the eigenspace corresponding to each eigenvalue. What property can you observe about the eigenvectors if $\mathbf{A}$ is Hermitian? Recall that the dot product is given by $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \overline{\mathbf{y}}$ for complex matrices.

Problem #2

The center of mass of a three dimensional body is found with the following formulas

$$x_c = \frac{\int_V x\gamma(x,y,z)dV}{\int_V \gamma(x,y,z)dV}, y_c = \frac{\int_V y\gamma(x,y,z)dV}{\int_V \gamma(x,y,z)dV}, z_c = \frac{\int_V z\gamma(x,y,z)dV}{\int_V \gamma(x,y,z)dV}$$

where $\gamma(x,y,z)$ is the density. Find the coordinates of the center of mass for the top half (i.e. limited by z=0) of a sphere of radius R centered at the origin. Assume that the density $\gamma(x,y,z) = \gamma_o$ is uniform.

Problem #3

Consider the following series and state if they will diverge or converge

a) $1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\ldots+\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot\ldots\cdot n}+\ldots$

b) $\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+\ldots+\frac{2^n}{n}+\ldots$

Problem 4.

a) What is the general solution of the following ODE?

$$\frac{dy}{dx} = -2xy + 3$$

b) What is the solution to the following initial value problem?

$$\frac{dy}{dx} = \frac{2y}{x} + x^4, \; y(1) = -6$$

c) What is the solution to the following initial value problem?

$$\frac{dy}{dx} = \frac{x^2}{e^y + \cos y}, \; y(-1) = 0$$

d) What is the solution to the following initial value problem?

$$x^2 \frac{dy}{dx} + 2xy = 0, \; y(2) = -3$$

e) What is the solution to the following initial value problem?

$$\frac{dy}{dx} = \frac{2y^3 + x^2 y}{x^3}, \; y(1) = 2$$

Problem 5.

The concentration c of phytoplankton in the upper ocean is described by the equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial z}\left(k\frac{\partial c}{\partial z}\right)$$

where z is the distance below the surface and k is a diffusion coefficient.

i) Formulate the boundary condition at $z = 0$ for this problem.

ii) If c is uniform for all z at $t = 0$ and $k = 6\ \mathrm{m}^2/\mathrm{day}$, find an expression for the phytoplankton concentration field as a function of time.

Problem 6.

The elevation ϕ of a thin membrane stretched over a wire loop in the (x, y) plane is described by the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in the square region $0 < x < 1$, $0 < y < 1$, where $\phi = 0$ everywhere on the boundary of this region except on the segment $0 < x < 1$, $y = 0$, where $\phi = x(1 - x)$.

i) Sketch the expected form of the solution as contours of equal values of ϕ.

ii) Develop your solution to the point at which the only part remaining is to satisfy the boundary condition at $y = 0$.

iii) Explain how a complete solution would be developed that satisfied this boundary condition.