Department of Mechanical Engineering
Michigan State University
Mathematics
Ph.D. Qualifying Examination
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One Book, Closed Notes
All six questions are weighted equally
Question 7 is for extra credit

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Question 1

Solve the initial value problem

\[ \frac{d x_1}{dt} + 2x_1 - x_2 = 1 + e^{-t} \]

\[ \frac{d x_2}{dt} + x_1 + 2x_2 = 3 \]

with

\[ x_1(0) = \frac{5}{2} \text{ and } x_2(0) = -\frac{1}{2} \]
Questions 2

Determine whether the vectors $v_1 = (1,1,0)$, $v_2 = (1,-2,2)$, and $v_3 = (3,0,3)$ are linearly independent.

If they are not linearly independent, give the relation among the three vectors and suggest a new vector or vectors which, along with $v_1$, $v_2$ and $v_3$, span $\mathbb{R}^3$.

If they are linearly independent, express the vector $(1,1,1)$ as a linear combination of $v_1$, $v_2$ and $v_3$. 

Question 3

Derive the Taylor series of $\frac{1}{x-1}$ about $x=4$ and determine the radius of convergence $R$ of the Taylor series expansion.
Question 4

What is the best linear approximation \( g(x) \) to the function \( f(x) = 2x^3 + 4x^2 - 2 \) in the interval \([-1, 1]\)? Use a least square approach, i.e., define \( g \) to minimize the error \( \int_{-1}^{1} (f - g)^2 \, dx \).
Question 5

Solve the diffusion equation

\[ \frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 \leq x \leq L, \quad t > 0 \]

with the boundary conditions

\[ u(0,t) = u(L,t) = 0, \quad t > 0 \]

and the initial condition

\[ u(x,0) = \begin{cases} 
  x, & 0 \leq x \leq L/2 \\
  L - x, & L/2 \leq x \leq L 
\end{cases} \]
Questions 6

Diagonalize the matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

i.e., find matrices \(Q\) and \(D\) such that \(Q^{-1}AQ = D\) and \(D\) is diagonal. If \(A\) cannot be diagonalized, explain why.