Student Code Number_________

Department of Mechanical Engineering
Michigan State University

Mathematics
Ph.D. Qualifying Exam

August, 2021

• Closed book, notes and a non-networked calculator
• All four questions are weighted equally.
• All work and all steps must be shown, if all work is not provided credit will not be given.

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NOTE: LAPLACE TABLES PROVIDED
Possibly useful information for problems 1 and 2:

1. See the attached Laplace Transform tables:

2. Table 1. Particular-solution forms for constant-coefficient linear ODEs:

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$y_p(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 0$ is not a root of the characteristic equation</td>
<td>$y_p(x) = A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n$</td>
</tr>
<tr>
<td>$m = 0$ is a root</td>
<td>$y_p(x) = t(A_0 + A_1 t + A_2 t + \ldots + A_n t^n)$</td>
</tr>
<tr>
<td>$m = 0$ is a repeated root</td>
<td>$y_p(x) = t^2 (A_0 + A_1 t + A_2 t^2 + \ldots + A_n t^n)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f(t) = Ce^{kt}$</th>
<th>$y_p(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = k$</td>
<td>$y_p(t) = Ae^{kt}$</td>
</tr>
<tr>
<td>$m = k$ is a root</td>
<td>$y_p(t) = Axe^{kt}$</td>
</tr>
<tr>
<td>$m = k$ is a repeated root</td>
<td>$y_p(t) = At^2 e^{kt}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$g(t) = C\cos(kt)$</th>
<th>$y_p(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = ik$ is not a root</td>
<td>$y_p(t) = A\cos(kt) + B\sin(kt)$</td>
</tr>
<tr>
<td>$m = ik$ is a root</td>
<td>$y_p(t) = t(A\cos(kt) + B\sin(kt))$</td>
</tr>
</tbody>
</table>

3. Fourier Series:

Fourier series of a periodic signal with period $2T$ is given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{n\pi t}{T} \right) + b_n \sin \left( \frac{n\pi t}{T} \right) \right),$$

where, the expansion coefficients $a_0$, $a_n$, and $b_n$ can be obtained using:

$$a_0 = \frac{1}{T} \int_{-T}^{T} f(t) dt,$$

$$a_n = \frac{1}{T} \int_{-T}^{T} f(t) \cos \left( \frac{n\pi t}{T} \right) dt,$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(t) \sin \left( \frac{n\pi t}{T} \right) dt, \quad n = 1,2,3, \ldots$$

4. Partial Fractions:

- Let $Q(s)$ have $m$ unrepeat ed real roots: $a_1, a_2, \ldots, a_m$. Thus,

$$Q(s) = (s - a_1)(s - a_2) \ldots (s - a_m),$$

then,

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \ldots + \frac{A_m}{s-a_m}.$$

Where,

$$A_i = \lim_{s \to a_i} (s - a_i)F(s)$$
Let \( Q(s) \) have \( n \) roots \( a_1, a_2, a_3, \ldots, a_n \), with \( a_1 \) repeated \( m \) times, then
\[
F(s) = \frac{P(s)}{Q(s)} = \frac{B_m}{(s-a_1)^m} + \cdots + \frac{B_2}{(s-a_2)^2} + \frac{B_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n},
\]
where,
\[
B_m = \lim_{s \to a_1} \frac{P(s)}{Q(s)} (s - a_1)^m,
B_{m-1} = \lim_{s \to a_1} \frac{d}{ds} \left[ \frac{P(s)}{Q(s)} (s - a_1)^m \right],
B_{m-2} = \frac{1}{2} \lim_{s \to a_1} \frac{d^2}{ds^2} \left[ \frac{P(s)}{Q(s)} (s - a_1)^m \right],
B_{m-3} = \frac{1}{3 \times 2} \lim_{s \to a_1} \frac{d^3}{ds^3} \left[ \frac{P(s)}{Q(s)} (s - a_1)^m \right],
\]
\[
\vdots
\]
\[
B_1 = \frac{1}{(m-1)!} \lim_{s \to a_1} \frac{d^{m-1}}{ds^{m-1}} \left[ \frac{P(s)}{Q(s)} (s - a_1)^m \right].
\]

Possibly useful information & formulas for Problems 3 and 4:

1. **Gradient:** Also known as the “del” operator, \( \nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \)

2. **Partial derivative:** If \( u(x, y) \) then \( \frac{\partial u}{\partial x} \) means taking the derivative of \( u(x, y) \) with respect to \( x \) while holding \( y = constant \).

3. **Total derivative of a field function:** \( du(x, y, z, t) = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz + \frac{\partial u}{\partial t} dt \).

4. **Partial differential equation:** A mathematical relation between quantities that are differentiated with respect to several (at least two) different independent variables.

5. **Fourier Series:**
   
   (1) Sine series: \( f(t) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n \pi t}{T} \right) ; \quad b_n = \frac{1}{T} \int_{-T}^{T} f(s) \sin \left( \frac{n \pi s}{T} \right) ds, \quad -T < t < T. \)
   
   (2) Cosine series: \( f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n \pi t}{T} \right) ; \quad a_n = \frac{1}{T} \int_{-T}^{T} f(s) \cos \left( \frac{n \pi s}{T} \right) ds, \quad -T < t < T. \)

The **general series** is given in the information for Problems 1 and 2 above.
6. **Trigonometric Identities:**

   (1) \( \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \)

   (2) \( \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \)

7. **Superposition Principle:** For a linear PDE the solution \( u(x, y, z, t) \equiv u(\vec{x}, t) \) can be written as a superposition of solutions \( u_1(\vec{x}, t), u_2(\vec{x}, t), \ldots \) as needed, each component satisfying different BCs.

8. **Partial Derivative:** If \( f(x, y, z) \), the differentiation operation \( \frac{\partial f}{\partial x} \) holds \( y, z \) constant, \( \frac{\partial f}{\partial y} \) holds \( x, z \) constant, \( \frac{\partial f}{\partial z} \) holds \( x, y \) constant.

9. **Gram-Schmidt Orthogonalization:** \( v_i = u_i - \sum_{j=1}^{i-1} \frac{u_i \cdot v_j}{v_j \cdot v_j} v_j \), where the \( v_i \) and the \( u_i \) are vectors.

10. **Separation of Variables:** For the solution of a partial differential equation for the function \( \varphi(x_1, x_2, x_3, t) \) we try \( \varphi(x_1, x_2, x_3, t) = \Phi_1(x_1) \Phi_2(x_2) \Phi_3(x_3) \Phi_4(t) \). If this solution works the equations for the component functions \( \Phi_i \) will be ODEs. In addition, the BC and ICs will sort out properly and the problem will be well-posed.
Problem 1: ODE with Periodic Forcing:

Find the general solution of the following ODE:

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 16y = \frac{1}{2}F(t),$$

where $F(t)$ is a periodic function $F(t) = F(t + 2\pi)$, and

$$F(t) = \begin{cases} 
  t; & -\frac{\pi}{2} < t < \frac{\pi}{2} \\
  \pi - t; & \frac{\pi}{2} < t < 3\frac{\pi}{2}
\end{cases}$$
Problem 2: ODE with Non-Periodic Forcing:

If $F(t)$ in Problem 1 is only active for the first period $0 \leq t \leq 2\pi$ and is zero otherwise, solve Problem 1 using Laplace Transforms. Assume $y(0) = y'(0) = 0$, where the prime denotes differentiation with respect to time, $y' = \frac{dy}{dt}$. 
Problem 3: PDEs: Consider the two classes of partial differential equations:

\[ i \) \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \]

\[ ii) \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \]

The physical interval of interest is \( 0 < x < 1 \). The boundary conditions are

\[ u(0, t) = 1, u(1, t) = 1. \]

a) Without yet applying any initial condition or conditions, write, for each equation (i) and (ii) the general solution that satisfies the boundary conditions.

b) Now apply the IC \( u(x, 0) = 1 + 0.5sin2\pi x \) and use the appropriate equation in order to derive your final solution. Do not try to solve the problem for which this IC is a mismatch.

c) Draw the evolution in time of the final solution. Write the scales on the axes. Does it approach a steady state? Describe the solution.
Problem 4: Second Order System: In the vicinity of the stationary point \( x = y = 0 \), you are to:

a) Solve the initial value problem given by the equation(s)

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = A \begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
2 & 5 \\
-1 & -4
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix},
\]

subject to the initial condition(s)

\[
\begin{pmatrix}
x_0 \\
y_0
\end{pmatrix} = \begin{pmatrix}
3 \\
-2.999
\end{pmatrix}.
\]

b) Draw the solution field in the vicinity of the stationary point. Draw the solution in the \( x-y \) plane with \( x \) as abscissa (horizontal), \( y \) as ordinate (vertical). You do not need to calculate unit eigenvectors.

c) Orthogonalize the eigenvectors using Gram-Schmidt or any other method you are familiar with (if they are not already orthogonal).

d) Write the equivalent single-equation second order equations for both \( x(t) \) or \( y(t) \). You must include the appropriate initial conditions.