Student Code Number__________

Department of Mechanical Engineering
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Mathematics
Ph.D. Qualifying Exam

August, 2021

- Closed book, notes and a non-networked calculator
- All four questions are weighted equally.
- All work and all steps must be shown, if all work is not provided credit will not be given.

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NOTE: LAPLACE TABLES PROVIDED

Possibly useful information for problems 1 and 2:

1. **See the attached Laplace Transform tables:**

2. **Table 1. Particular-solution forms for constant-coefficient linear ODEs:**

$\boldsymbol{f(t)}$		$\boldsymbol{y_p(t)}$
Polynomial of degree n	$m = 0$ is not a root of the characteristic equation	$y_p(x) = A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n$
	$m = 0$ is a root	$y_p(x) = t(A_0 + A_1 t + A_2 t + \cdots + A_n t^n)$
	$m = 0$ is a repeated root	$y_p(x) = t^2(A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n)$
$f(t) = Ce^{kt}$	$m = k$	$y_p(t) = Ae^{kt}$
	$m = k$ is a root	$y_p(t) = Axe^{kt}$
	$m = k$ is a repeated root	$y_p(t) = At^2 e^{kt}$
$g(t) = C\cos(kt)$	$m = ik$ is not a root	$y_p(t) = A\cos(kt) + B\sin(kt)$
	$m = ik$ is a root	$y_p(t) = t(A\cos(kt) + B\sin(kt))$

3. **Fourier Series:**

Fourier series of a periodic signal with period $2T$ is given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right)\right),$$

where, the expansion coefficients a_0, a_n, and b_n can be obtained using:

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(t)dt,$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(t)\cos\left(\frac{n\pi t}{T}\right)dt,$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(t)\sin\left(\frac{n\pi t}{T}\right)dt,\ \ n = 1,2,3,\ldots$$

4. **Partial Fractions:**

- Let $Q(s)$ have m unrepeated real roots: $a_1, a_2, \ldots, a_m$. Thus,

$$Q(s) = (s - a_1)(s - a_2)\ldots(s - a_m),$$

then,

$$F(s) = \frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_m}{s-a_m}.$$

Where,

$$A_i = \lim_{s \to a_i}(s - a_i)F(s)$$

- Let $Q(s)$ have n roots $a_1, a_2, a_3, \dots, a_n$, with a_1 repeated m times, then
$$F(s) = \frac{P(s)}{Q(s)} = \frac{B_m}{(s-a_1)^m} + \cdots + \frac{B_2}{(s-a_1)^2} + \frac{B_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n}.$$

where,

$$B_m = \lim_{s \to a_1} \frac{P(s)}{Q(s)} (s-a_1)^m,$$
$$B_{m-1} = \lim_{s \to a_1} \frac{d}{ds} \left[\frac{P(s)}{Q(s)} (s-a_1)^m \right],$$
$$B_{m-2} = \frac{1}{2} \lim_{s \to a_1} \frac{d^2}{ds^2} \left[\frac{P(s)}{Q(s)} (s-a_1)^m \right],$$
$$B_{m-3} = \frac{1}{3\times 2} \lim_{s \to a_1} \frac{d^3}{ds^3} \left[\frac{P(s)}{Q(s)} (s-a_1)^m \right],$$

⋮

$$B_1 = \frac{1}{(m-1)!} \lim_{s \to a_1} \frac{d^{m-1}}{ds^{m-1}} \left[\frac{P(s)}{Q(s)} (s-a_1)^m \right].$$

Possibly useful information & formulas for Problems 3 and 4:

1. **Gradient:** Also known as the "del" operator, $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$.

2. **Partial derivative:** If $u(x,y)$ then $\frac{\partial u}{\partial x}$ means taking the derivative of $u(x,y)$ with respect to x while holding $y = constant$.

3. **Total derivative of a field function:** $du(x,y,z,t) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial t}dt$.

4. **Partial differential equation:** A mathematical relation between quantities that are differentiated with respect to several (at least two) different independent variables.

5. **Fourier Series:**

(1) Sine series: $f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T}\right)$; $b_n = \frac{1}{T}\int_{-T}^{T} f(s) \sin\left(\frac{n\pi s}{T}\right) ds$, $-T < t < T$.

(2) Cosine series: $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{T}\right)$; $a_n = \frac{1}{T}\int_{-T}^{T} f(s) \cos\left(\frac{n\pi s}{T}\right) ds$, $-T < t < T$.

The **general series** is given in the information for Problems 1 and 2 above.

6. **Trigonometric Identities:**

(1) $\sin(A \pm B) = sinAcosB \pm cosAsinB$

(2) $\cos(A \pm B) = cosAcosB \ \mp sinAsinB$

7. **Superposition Principle:** For a linear PDE the solution $u(x, y, z, t) \equiv u(\vec{x}, t)$ can be written as a *superposition of solutions* $u_1(\vec{x}, t)$, $u_2(\vec{x}, t), \ldots$ as needed, each component satisfying different BCs.

8. **Partial Derivative:** If *f(x, y, z)*, the differentiation operation *∂f/∂x* holds *y*, *z* constant, *∂f/∂y* holds *x*, *z* constant, *∂f/∂z* holds *x*, *y* constant.

9. **Gram-Schmidt Orthogonalization:** $v_i = u_i - \sum_{j=1}^{i-1} \frac{u_i \cdot v_j}{v_j \cdot v_j} v_j$, where the v_i and the u_i are vectors.

10. **Separation of Variables:** For the solution of a partial differential equation for the function $\varphi(x_1, x_2, x_3, t)$ we try $\varphi(x_1, x_2, x_3, t) = \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(t)$. If this solution works the equations for the component functions Φ_i will be ODEs. In addition, the BC and ICs will sort out properly and the problem will be *well-posed*.

Problem 1: ODE with Periodic Forcing:

Find the general solution of the following ODE:

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 16y = \frac{1}{2}F(t),$$

where $F(t)$ is a periodic function $F(t) = F(t + 2\pi)$, and

$$F(t) = \begin{cases} t; & -\dfrac{\pi}{2} < t < \dfrac{\pi}{2} \\ \pi - t; & \dfrac{\pi}{2} < t < 3\dfrac{\pi}{2} \end{cases}$$

Problem 2: ODE with Non-Periodic Forcing:

If $F(t)$ in **Problem 1** is only active for the first period $0 \leq t \leq 2\pi$ and is zero otherwise, solve **Problem 1** using Laplace Transforms. Assume $y(0) = y'(0) = 0$, where the prime denotes differentiation with respect to time, $y' = \frac{dy}{dt}$.

Problem 3: PDEs: Consider the two classes of partial differential equations:

$$i)\ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

$$ii)\ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

The physical interval of interest is $0 < x < 1$. The boundary conditions are

$$u(0,t) = 1, u(1,t) = 1.$$

a) Without yet applying any initial condition or conditions, write, for each equation (i) and (ii) the general solution that satisfies the boundary conditions.
b) Now apply the IC $u(x,0) = 1 + 0.5sin2\pi x$ and use the appropriate equation in order to derive your final solution. Do not try to solve the problem for which this IC is a mismatch.
c) Draw the evolution in time of the final solution. Write the scales on the axes. Does it approach a steady state? Describe the solution.

Problem 4: Second Order System: In the vicinity of the stationary point $x = y = 0$, you are to:

a) Solve the initial value problem given by the equation(s)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \dot{\vec{x}} = \vec{A}\vec{x} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

subject to the initial condition(s)

$$\begin{pmatrix} x_o \\ y_o \end{pmatrix} = \begin{pmatrix} 3 \\ -2.999 \end{pmatrix}.$$

b) Draw the solution field in the vicinity of the stationary point. Draw the solution in the *x*-*y* plane with *x* as abscissa (horizontal), *y* as ordinate (vertical). You do not need to calculate unit eigenvectors.

c) Orthogonalize the eigenvectors using Gram-Schmidt or any other method you are familiar with (if they are not already orthogonal).

d) Write the equivalent single-equation second order equations for *both* $x(t)$ or $y(t)$. You must include the appropriate initial conditions.