Math Qualifying Exam
Department of Mechanical Engineering

January 2020

Closed book, closed notes and a non-networked calculator

You may bring a one page (8.5×11) one sided formula sheet

All work and steps must be shown. If work is not shown, then credit will not be given

All problems have equal weight

Laplace Transform Tables are provided

1. Consider the following ODE on $t \geq 0$ with initial conditions as follows:

$$\ddot{y} - 3\dot{y} + 2y = f(t), \qquad y(0) = 1, \quad \dot{y}(0) = 0,$$

with

$$f(t) = \begin{cases} 0, & \text{for} \quad 0 < t < 5, \\ 2, & \text{for} \quad 5 < t < 10, \\ -1, & \text{for} \quad 10 < t < 15, \\ 0, & \text{for} \quad t > 15. \end{cases}$$

Carefully write $f(t)$ with the aid of the unit step function U, i.e.,

$$f(t) = 2\,U(t-5) + \ldots$$

and then use Laplace transforms to solve the ODE. Note, methods that do not use the Laplace transform will not receive credit.

2. Let

$$\mathsf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathsf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}, \qquad \mathsf{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \qquad \mathsf{v}_4 = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}, \qquad \mathsf{b} = \begin{bmatrix} 5 \\ -4 \\ 7 \end{bmatrix}.$$

- (a) Consider the collection $\{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3\}$ and also the collection $\{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_4\}$. One of these is a collection of linearly dependent vectors and the other is a collection of linearly independent vectors. Determine which collection is linearly independent.
- (b) Using the linearly independent collection from part (a), express b as a linear combination of the vectors in that collection.

3. Find the eigenvalues and eigenvectors of the rotation matrix

$$\mathsf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

4. Develop an infinite series solution $u(x, y)$ to the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{y^2}\frac{\partial u}{\partial y} + 8y\,u = 0$$

for $y > H$ and $0 < x < L$. The function $u(x, y)$ is subject to boundary conditions:

$$\frac{\partial u}{\partial x} = 0, \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L.$$

Specifically, develop a formal solution in the form

$$u(x, y) = \sum_n X_n(x)\, Y_n(y)$$

with specific expressions for $X_n(x)$ and $Y_n(y)$.

APPENDIX III

Table of Laplace Transforms

	$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1.	1	$\dfrac{1}{s}$
2.	t	$\dfrac{1}{s^2}$
3.	t^n	$\dfrac{n!}{s^{n+1}}$, n a positive integer
4.	$t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$
5.	$t^{1/2}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$
6.	t^{α}	$\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$
7.	$\sin kt$	$\dfrac{k}{s^2+k^2}$
8.	$\cos kt$	$\dfrac{s}{s^2+k^2}$
9.	$\sin^2 kt$	$\dfrac{2k^2}{s(s^2+4k^2)}$
10.	$\cos^2 kt$	$\dfrac{s^2+2k^2}{s(s^2+4k^2)}$
11.	e^{at}	$\dfrac{1}{s-a}$
12.	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
13.	$\cosh kt$	$\dfrac{s}{s^2-k^2}$

14. $\sinh^2 kt$	$\dfrac{2k^2}{s(s^2-4k^2)}$
15. $\cosh^2 kt$	$\dfrac{s^2-2k^2}{s(s^2-4k^2)}$
16. $e^{at}t$	$\dfrac{1}{(s-a)^2}$
17. $e^{at}t^n$	$\dfrac{n!}{(s-a)^{n+1}}$, n a positive integer
18. $e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
19. $e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
20. $e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$
21. $e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$
22. $t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
23. $t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
24. $\sin kt + kt\cos kt$	$\dfrac{2ks^2}{(s^2+k^2)^2}$
25. $\sin kt - kt\cos kt$	$\dfrac{2k^3}{(s^2+k^2)^2}$
26. $t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
27. $t\cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$
28. $\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$
29. $\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
30. $1-\cos kt$	$\dfrac{k^2}{s(s^2+k^2)}$
31. $kt-\sin kt$	$\dfrac{k^3}{s^2(s^2+k^2)}$
32. $\cos at - \cos bt$	$\dfrac{s(b^2-a^2)}{(s^2+a^2)(s^2+b^2)}$
33. $\sin kt \sinh kt$	$\dfrac{2k^2s}{s^4+4k^4}$
34. $\sin kt \cosh kt$	$\dfrac{k(s^2+2k^2)}{s^4+4k^4}$

35. $\cos kt \sinh kt$	$\dfrac{k(s^2 - 2k^2)}{s^4 + 4k^4}$
36. $\cos kt \cosh kt$	$\dfrac{s^3}{s^4 + 4k^4}$
37. $\delta(t)$	1
38. $\delta(t - a)$	e^{-as}
39. $\mathscr{U}(t - a)$	$\dfrac{e^{-as}}{s}$
40. $J_0(kt)$	$\dfrac{1}{\sqrt{s^2 + k^2}}$
41. $\dfrac{e^{bt} - e^{at}}{t}$	$\ln \dfrac{s - a}{s - b}$
42. $\dfrac{2(1 - \cos at)}{t}$	$\ln \dfrac{s^2 + a^2}{s^2}$
43. $\dfrac{2(1 - \cosh at)}{t}$	$\ln \dfrac{s^2 - a^2}{s^2}$
44. $\dfrac{\sin at}{t}$	$\arctan\left(\dfrac{a}{s}\right)$
45. $\dfrac{\sin at \cos bt}{t}$	$\dfrac{1}{2} \arctan \dfrac{a + b}{s} + \dfrac{1}{2} \arctan \dfrac{a - b}{s}$
46. $\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
47. $\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
48. $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$
49. $2\sqrt{\dfrac{t}{\pi}}\, e^{-a^2/4t} - a \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
50. $e^{ab}e^{b^2t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s} + b)}$
51. $-e^{ab}e^{b^2t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{be^{-a\sqrt{s}}}{s(\sqrt{s} + b)}$
52. $e^{at}f(t)$	$F(s - a)$
53. $f(t - a)\mathscr{U}(t - a)$	$e^{-as}F(s)$
54. $g(t)\mathscr{U}(t - a)$	$e^{-as}\mathscr{L}\{g(t + a)\}$
55. $f^{(n)}(t)$	$s^nF(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$
56. $t^nf(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
57. $\int_0^t f(\tau)g(t - \tau)\, d\tau$	$F(s)G(s)$