Code No.______________________

Mathematics Ph.D Qualifier Exam

Department of Mechanical Engineering
Michigan State University

August 2022

Directions: Closed book, one sheet of your own notes permitted

All problems carry equal weight.

Exam prepared by Profs. G. Brereton and D. Segalman

Problem 1.

For the ordinary differential equation

$$\frac{d^2 f}{dx^2} + f = \sin x + 1$$

a) find the general solution;

b) find a particular solution; and

c) find the specific solution consistent with the conditions $f(0) = 0$ and $f(\pi/2) = 2$.

Problem 2.

A rod of length L carries an electric current that causes Ohmic heating, described by a uniform heat source S. Its local temperature $\Theta(x,t)$ is described by the equation and conditions:

$$\frac{\partial \Theta}{\partial t} = \kappa \frac{\partial^2 \Theta}{\partial x^2} + S \,, \quad \Theta(x,0) = 0 \,, \quad \Theta(0,t) = 0 \,, \quad \Theta(L,t) = 0$$

where $S = 1\,°\mathrm{C/sec}$, $\kappa = 10^{-4}\ \mathrm{m^2/s}$, and $L = 2$ m.

a) find the complete solution $\Theta(x,t)$ to this problem;

b) use your answer to a) to determine the steady-state solution; and

c) find the steady-state solution a second way, by solving the given equation when $\partial\Theta/\partial t$ is set to 0.

Problem 3.

The triangular function shown has a period of 2. It can be described by

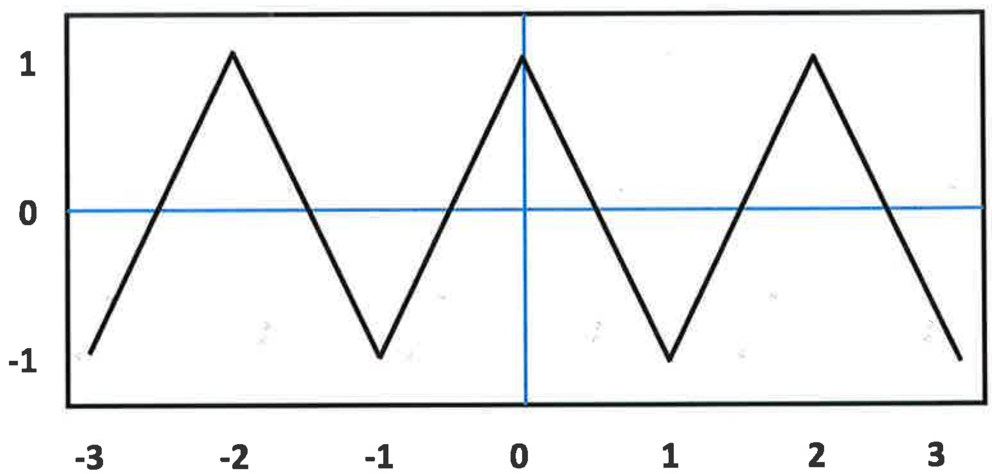
$$f(x) = 1 + 2x \quad \text{for} \quad -1 < x < 0 \quad \text{and} \quad f(x) = 1 - 2x \quad \text{for} \quad 0 < x < 1$$

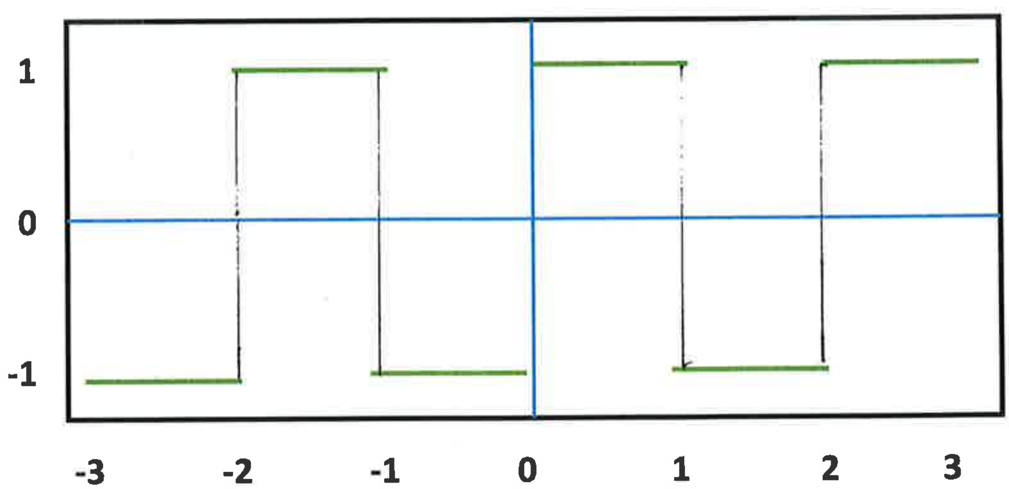
a) Find the Fourier series for $f(x)$.

b) Use your answer to a) to find the Fourier series for the square-wave function shown below, and described by

$$f(x) = -1 \quad \text{for} \quad -1 < x < 0 \quad \text{and} \quad f(x) = 1 \quad \text{for} \quad 0 \leq x < 1$$

A table of some integrals that may be useful is provided at the end of the exam booklet.





Problem 4.

Find the first three terms of the Taylor series of the given functions about $z = a$.

a)

$$f(z) = \frac{1}{1-z} \ , \quad a = 3i$$

b)

$$f(z) = \tan z \ , \quad a = \frac{\pi}{4}$$

c)

$$f(z) = \frac{2-3z}{2z^2-3z+1} \ , \quad a = -1$$

Problem 5.

The reaction rates of chemicals x and y in a chemical reaction are described by:

$$\frac{dx}{dt} = 2y + 1 \tag{1}$$

$$\frac{dy}{dt} = -8x \tag{2}$$

with $x(0) = 0$, $y(0) = 2$. Using Laplace transformation, find $x(t)$ and $y(t)$.

If you found $x(t)$ and $y(t)$ by inverse transforming $X(s)$ and $Y(s)$, substitute $x(t)$ and $y(t)$ into either (1) or (2) to show your answers are correct.

If you found one of $x(t)$ and $y(t)$ by inverse transforming, and the other from (1) (or (2)), substitute $x(t)$ and $y(t)$ into (2) (or (1)) to show your answers are correct.

A table of Laplace transforms is provided at the end of the exam booklet.

Problem 6.

a) Find the inverse of the matrix

$$\begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}$$

b) Find the eigenvalues and eigenvectors of the matrix shown.

c) Can this matrix be diagonalized by multiplication by a transformation matrix? If yes, explain briefly why. If no, explain briefly why.

$\int \cos^2(ax)dx = \frac{\sin(2ax)}{4a} + \frac{x}{2} + C$	$\int \sin^2(ax)dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C$
$\int \sin(ax)\cos(ax)dx = -\frac{\cos^2(ax)}{2a} + C$	
$\int x\cos(ax)dx = \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$	$\int x\sin(x)dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$

Table 1: Table of some relevant integrals

Function	**Laplace Transform**
$f(t)$	$F(s) = \int_0^\infty f(t)\exp(-st)dt$
t^n	$n!/s^{n+1}$
$\sin kt$	$k/(s^2+k^2)$
$\cos kt$	$s/(s^2+k^2)$
$\exp(at)$	$1/(s-a)$
$f*g = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
dy/dt	$sY(s)-y(0)$
d^2y/dt^2	$s^2Y(s)-sy(0)-\dot{y}(0)$
$f(t)U(t-a)$	$\exp(-as)\mathcal{L}\{f(t+a)\}$
$af(t)+bg(t)$ (a, b are constants)	$aF(s)+bG(s)$
$f(t-a)U(t-a)$	$\exp(-as)F(s)$
$\exp(at)f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n d^nF(s)/ds^n$