Math Qualifying Exam
Department of Mechanical Engineering

January 2018

Open book (only one book allowed)
Open notes (limited to one 3 ring binder of notes in the student’s own handwriting - no photocopies)
In order to receive full credit, you must show all work
Calculators NOT permitted

Answer all questions
All questions have equal weight

TIME: 3.0 hrs

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Problem 1
Find all the eigenvalues of the following matrix. Then find an eigenvector of length one corresponding to the largest eigenvalue.

\[
\begin{bmatrix}
2 & 0 & 2 \\
0 & 2 & 0 \\
3 & 0 & 1
\end{bmatrix}
\]
Problem 2
Use Taylor series expansion to show that

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]

Then show that

\[ (1 + x) e^x = \sum_{n=0}^{\infty} \frac{n + 1}{n!} x^n \]
Problem 3
A is a 3x3 matrix such that $A^T = -A$. Prove that the determinant of $A$ is zero.
Problem 4
Consider an object totally immersed in a water bath. There is an upward buoyant force on the body due to pressure of the fluid on the body's surface. At any point \( P = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \) on the surface the traction there is \( \sigma(P) = -p(P)\mathbf{n}(P) \) where \( \mathbf{n}(P) \) is a unit vector pointing outward from the body and the pressure is \( p(P) = -\rho g z \). Density \( \rho \) and gravity \( g \) are constant numbers. Calculate the buoyant force due to the fluid pressure in the following steps:

1. Sketch the problem showing the body, a typical point \( P \) on its surface, the corresponding normal vector \( \mathbf{n}(P) \) and coordinate directions. (Let \( \mathbf{k} \) be up.)
2. State the divergence theorem.
3. From the divergence theorem show that for any differentiable scalar field \( g(P) \)

\[
\int_S g(P)\mathbf{n}(P)dS = \int_V \nabla g(P)dV.
\]
4. Using the previous step, show that the resultant buoyancy force is \( \mathbf{F} = \rho gV\mathbf{k} \) where \( V \) is the volume of the body.
Problem 5
Two differential equations are given below. One is exact and one is not.
Find the equation of a curve obeying the exact equation that passes through
the origin. When x=1 on this curve, what is y? (Your value or values of y
are allowed to contain square roots).

(i) \( (6y - 5x) \, dx + (y+6x) \, dy = 0, \) 
(ii) \( (2y+3x) \, dx + (3y-8x) \, dy = 0. \)
Problem 6
Consider the complex function $f(z) = (z-5) / (a z + 2i)$ where $a$ in the denominator is a complex number that needs to be determined. If $a$ is such that $f(i) = i$, then find $f(1)$ and $f(-i)$, expressing each in the form $b + c i$ where $b$ and $c$ are real. Show all work!