Math Qualifying Exam
Department of Mechanical Engineering

January 2018

Open book (only one book allowed)
Open notes (limited to one 3 ring binder of notes in the student's own handwriting - no photocopies)
In order to receive full credit, you must show all work
Calculators NOT permitted

Answer all questions
All questions have equal weight

TIME: 3.0 hrs

Prepared by: Profs. S. Baek, Thomas Pence and Dan Segalman

Problem 1

Find all the eigenvalues of the following matrix. Then find an eigenvector of length one corresponding to the largest eigenvalue.

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Problem 2

Use Taylor series expansion to show that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Then show that

$$(1+x)\,e^x = \sum_{n=0}^{\infty} \frac{n+1}{n!} x^n .$$

Problem 3

A is a 3x3 matrix such that $A^T = -A$. Prove that the determinant of A is zero.

Problem 4

Consider an object totally immersed in a water bath. There is an upward buoyant force on the body due to pressure of the fluid on the body's surface. At any point $P = x\vec{i} + y\vec{j} + z\vec{k}$ on the surface the traction there is $\sigma(P) = -p(P)\vec{n}(P)$ where $\vec{n}(P)$ is a unit vector pointing outward from the body and the pressure is $p(P) = -\rho g z$. Density ρ and gravity g are constant numbers. Calculate the buoyant force due to the fluid pressure in the following steps:

1. Sketch the problem showing the body, a typical point P on its surface, the corresponding normal vector $\vec{n}(P)$ and coordinate directions. (Let $\vec{k}$ be up.)
2. State the divergence theorem.
3. From the divergence theorem show that for any differentiable scalar field $g(P)$

$$\int_S g(P)\vec{n}(P)\,dS = \int_V \nabla g(P)\,dV.$$

4. Using the previous step, show that the resultant buoyancy force is $\vec{F} = \rho g V \vec{k}$ where V is the volume of the body.

Problem 5

Two differential equations are given below. One is exact and one is not. Find the equation of a curve obeying the exact equation that passes through the origin. When $x=1$ on this curve, what is y? (Your value or values of y are allowed to contain square roots).

(i) $(6y - 5x)\,dx + (y+6x)\,dy = 0$, (ii) $(2y+3x)\,dx + (3y-8x)\,dy = 0$.

Problem 6

Consider the complex function $f(z) = (z-5) / (a z + 2i)$ where a in the denominator is a complex number that needs to be determined. If a is such that $f(i) = i$. then find $f(1)$ and $f(-i)$, expressing each in the form $b + c i$ where b and c are real. Show all work!