Fluid Mechanics Qualifying Exam
Department of Mechanical Engineering

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Closed book, but formula sheets are provided.

All questions have the same weight.

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Problem 1. The flow illustrated in the figure is given by 
\( u = A(x^2 - y^2), \ v = -2Ax, \ w = 0, \) where \( u, v \) and \( w \) are velocity components and \( A \) is a constant. Check if this is an irrotational flow. If so, find the velocity potential and demonstrate that it satisfies Laplace’s equation \( \nabla^2 \phi = 0. \) Also, check to see if the continuity equation is satisfied.
Problem 2. A liquid column of total length $L$ oscillates in a U-tube as shown in the figure. Assuming frictionless flow, find an expression predicting the frequency of oscillation.
**Problem 3.** A two dimensional airfoil is placed in a constant area wind tunnel with height $h$ and width $W$. The incoming flow is uniform with contact velocity $V_1$ and pressure $P_1$. The velocity field downstream of the airfoil, $V_2$ is not uniform but may be assumed to linearly increase from zero at the centerline to $V_{2m}$ close to wall. The flow pressure at this downstream location, $P_2$ is lower than $P_1$. The pressure at top and bottom surfaces of the channel are $P_t(x)$ and $P_b(x)$, respectively. Compute the lift and drag forces generated by the airfoil as a function of $V_1$, $P_1$, $P_2$, $P_t(x)$, $P_b(x)$, $h$, $W$. What are the lift and drag coefficients? (hint: lift and drag coefficients are lift and drag forces normalized by the dynamic pressure). Assume the flow streamlines at stations 1 and 2 to be parallel and neglect the viscous forces exerted by the tunnel walls on the fluid.
Problem 4. Simplify the compressible Navier-Stokes equations (attached continuity, momentum and energy equations) for a two-dimensional, steady, incompressible flow in the x-y plane. Ignore gravity. Can you simplify these equations further for a flat plat laminar boundary layer (FPLBL) by scaling? If yes, show how. Write the boundary conditions for the FPLBL flow. How do you solve the simplified equations for a FPLBL? (you do not need to solve them just state how).
Integral mass conservation equation:

\[ 0 = \frac{\partial}{\partial t} \int \rho dV + \int \rho \vec{V} \cdot d\vec{A} \]

Integral momentum equation:

\[ \vec{F} = \vec{F}_E + \vec{F}_g = \frac{\partial}{\partial t} \int \rho \vec{V} dV + \int \rho \vec{V} \cdot d\vec{A} \]

**The Equation of Continuity**

Rectangular Coordinates \((x, y, z)\):

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u_x) + \frac{\partial}{\partial y} (\rho u_y) + \frac{\partial}{\partial z} (\rho u_z) = 0 \]

Cylindrical Coordinates \((r, \theta, z)\):

\[ \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{\partial}{\partial z} (\rho u_z) = 0 \]

**Momentum Equations for Incompressible Newtonian Fluid with Constant Viscosity \((\mu)\)**

Rectangular Coordinates \((x, y, z)\):

\[
\rho \left( \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right) = \mu \left[ \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right] - \frac{\partial p}{\partial x} + \rho g_x
\]

\[
\rho \left( \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} \right) = \mu \left[ \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2} \right] - \frac{\partial p}{\partial y} + \rho g_y
\]

\[
\rho \left( \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} \right) = \mu \left[ \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right] - \frac{\partial p}{\partial z} + \rho g_z
\]

Cylindrical Coordinates \((r, \theta, z)\):

\[
\rho \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_x}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_y}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_x}{r} - \frac{u_y}{r} \frac{\partial u_r}{\partial \theta} \right) = \mu \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r} \frac{\partial u_r}{\partial \theta} \right] - \frac{\partial p}{\partial r} + \rho g_r
\]

\[
\rho \left( \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_x}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_y}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{u_x}{r} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{u_y}{r} \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) = \mu \left[ \frac{\partial^2 u_\theta}{\partial (r^2) \theta^2} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r} \frac{\partial u_\theta}{\partial \theta} \right] - \frac{\partial p}{\partial \theta} + \rho g_\theta
\]

\[
\rho \left( \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_x}{r} \frac{\partial u_z}{\partial \theta} + \frac{u_y}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) = \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right] - \frac{\partial p}{\partial z} + \rho g_z
\]

**Vorticity Equation in Rectangular Coordinates**:

\[
\vec{\Omega} = \vec{\Omega}_x \hat{i} + \vec{\Omega}_y \hat{j} + \vec{\Omega}_z \hat{k}
\]

\[
\vec{\Omega}_x = \frac{\partial V_y}{\partial z} - \frac{\partial V_z}{\partial y}
\]

\[
\vec{\Omega}_y = \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}
\]

\[
\vec{\Omega}_z = \frac{\partial V_x}{\partial y} - \frac{\partial V_y}{\partial x}
\]

**Bernoulli Equation For Compressible Unsteady Flows**:

\[
\rho \frac{\partial \vec{V}}{\partial t} \cdot ds + dp + \frac{1}{2} \rho d(V^2) + p g_z \cdot dz = 0 \quad \text{(along a streamline)}
\]

\[
V = \sqrt{\vec{V}^2} \quad \text{p = pressure} \quad \text{s = coordinate along streamline}
\]